Online Appendix to “Indexers and Comovement”

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A Proofs and model derivation

A.1 Setup

This section provides additional details on the model presented in Section 3. The basket of stocks 1 and 2 is called the index I, which by construction represents a value-weighted index, and the basket of stocks 1, 2 and 3 is called the market M. The index and market baskets therefore also pay dividend streams with dynamics as described in (1), with the exception that their variance parameters have the form:

$$\sigma_{D_I} = \left[1 - \frac{2s_1s_2}{(s_1+s_2)^2}(1-\rho_D)\right]\sigma_D^2, \tag{1}$$

$$\sigma_{D_M} = \left[1 - 2(s_1s_2 + s_1s_3 + s_2s_3)(1-\rho_D)\right]\sigma_D^2, \tag{2}$$

where s_i is the weight of share of dividends of asset i:

$$s_i = \frac{D_i}{D_1+D_2+D_3},\ i \in \{1,2,3\}. \tag{3}$$

Let $\omega_{i,t}$ denote the market weight of stock i at time t such that $\sum_{i=1}^{3}\omega_{i,t} = 1$ and let $\omega_{i,t}^I = \omega_{i,t}/(\omega_{1,t}+\omega_{2,t})$ denote the weight of asset $i \in \{1,2\}$ in the index. Then the index return moments are

$$\mu_{I,t} = \omega_{1,t}^I\mu_{1,t} + \omega_{2,t}^I\mu_{2,t}, \tag{4}$$

$$\sigma_{I,t}^2 = (\omega_{1,t}^I)^2\sigma_{1,t}^2 + (\omega_{2,t}^I)^2\sigma_{2,t}^2 + 2\omega_{1,t}^I\omega_{2,t}^I corr(dZ_{1,t}, dZ_{2,t})\sigma_{1,t}\sigma_{2,t}. \tag{5}$$

A.2 Agents' problem

Agent j's optimization problem at time t is to maximize her time additive utility:

$$U_{j,t} = E_t \left[\int_t^\infty e^{-\delta(s-t)} \log c_{j,s} ds \right] \tag{6}$$

subject to her budget constraint. Formally, this gives:

$$\max U_{j,t} \text{ subject to } \; E_t \left[\int_0^\infty \frac{\xi_{j,s}}{\xi_{j,t}} c_{j,s} ds \right] \leq W_{j,t}, \tag{7}$$

where $\xi_{j,t}$ is the marginal utility of agent j at time t. The first order condition is:

$$\kappa_j \frac{\xi_{j,s}}{\xi_{j,t}} = e^{-\delta(s-t)} c_{j,s}^{-1}, \tag{8}$$

where κ_j is the Lagrange multiplier on the budget constraint and $\xi_{j,t}$ is a process given by:

$$\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t}dt - \theta_{j,t}' dZ_t. \tag{9}$$

where $\theta_{j,t}$ is the price of risk process for agent j. Note that the process can also be written with respect to the dividend basis and the market basis[1] as:

$$\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t}dt - \overline{\theta}_{j,t}' d\overline{Z}_{D,t} = -r_{j,t}dt - \underline{\theta}_{j,t}' d\underline{Z}_t. \tag{10}$$

The rationale for using two different bases, in addition to the initial Brownian motions Z, is that each of the two new bases simplifies the derivation of the solution for a part of the problem and involves independent Brownian motions, which are easier to deal with. It is simpler to solve for optimal portfolios and market clearing under the market

[1] For a definition of the different bases, see Appendix B.

basis. However, the market basis transformation depends on stock return covariances, so it is not appropriate to solve for equilibrium price dynamics. The dividend basis is more useful for that purpose.

Since both agents trade in the bond, in equilibrium they should have the same riskless rate (i.e. $r_{\mathcal{I},t} = r_{\mathcal{A},t} = r_t$.) However their different investment opportunity sets means they will face different market price of risk. Following the convex duality methodology approach of Cvitanić and Karatzas (1992), I define a fictitious market which the indexer views as complete. In the current setup with log ulity, the market price of risk in the fictitious market is the same as in the incomplete market (see Example 7.2 on p.304 Karatzas and Shreve (1998) for more details.) The idea is to create a fictitious market for agent $\mathcal{I}$ by replacing the expected return on asset i by $\mu_i(\psi) = \mu_i + \psi_i$ such that in equilibrium she chooses not to hold the unavailable asset, and to hold the index assets according to index weights. In the present setup,

$$\psi = \text{argmin}_{\psi} \left[(\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r)\Sigma^{-1}(\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r)' \right]^{1/2}. \tag{11}$$

Substituting the ψ obtained in (11) in the shadow market price of risk of the indexer I obtain, under the market basis:

$$\underline{\theta}_{\mathcal{I}} = \phi_I \sigma_I^{-1} \begin{bmatrix} \sigma_1 \omega_1^I + \rho_{12}\sigma_2\omega_2^I \\ \sqrt{1-\rho_{12}^2}\sigma_2\omega_2^I \\ 0 \end{bmatrix}, \tag{12}$$

where $\phi_I = \frac{\mu_I - r}{\sigma_I}$ is the Sharpe ratio of the index. Since $(\sigma_1\omega_1^I + \rho_{12}\sigma_2\omega_2^I)^2 + (\sqrt{1-\rho_{12}^2}\sigma_2\omega_2^I)^2 = \sigma_I^2$, in scalar form $\theta_{\mathcal{I}} = \phi_I$. The result in (12) has the same form if working under the dividend basis following (10):

$$\overline{\theta}_{\mathcal{I}} = \phi_I \sigma_I^{-1} \begin{bmatrix} \omega_1^I \overline{\sigma}_{11} + \omega_2^I \overline{\sigma}_{21} \\ \omega_1^I \overline{\sigma}_{12} + \omega_2^I \overline{\sigma}_{22} \\ \omega_1^I \overline{\sigma}_{13} + \omega_2^I \overline{\sigma}_{23} \end{bmatrix}. \tag{13}$$

Agent $\mathcal{A}$ is unconstrained and faces complete markets, so her market price of risk under the market and dividend bases are given by:

$$\begin{aligned} \underline{\theta}_{\mathcal{A}} &= \underline{\sigma}^{-1}(\mu_1 - r,\ \mu_2 - r,\ \mu_3 - r)' \\ &= \begin{bmatrix} \phi_1 \\ \frac{\phi_1 - \rho_{12}\phi_2}{\sqrt{1-\phi_{12}^2}} \\ \frac{\phi_3(1-\rho_{12}^2) - \phi_1(\rho_{13}-\rho_{12}\rho_{23}) - \phi_2(\rho_{23}-\rho_{12}\rho_{13})}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}} \end{bmatrix}, \end{aligned} \tag{14}$$

$$\begin{aligned} \overline{\theta}_{\mathcal{A}} &= \overline{\sigma}^{-1}(\mu_1 - r,\ \mu_2 - r,\ \mu_3 - r)' \\ &= \frac{1}{c} \begin{bmatrix} x_1(\overline{\sigma}_{23}\overline{\sigma}_{32} - \overline{\sigma}_{22}\overline{\sigma}_{33}) + x_2(\overline{\sigma}_{12}\overline{\sigma}_{33} - \overline{\sigma}_{13}\overline{\sigma}_{32}) + x_3(\overline{\sigma}_{13}\overline{\sigma}_{22} - \overline{\sigma}_{12}\overline{\sigma}_{23}) \\ x_1(\overline{\sigma}_{21}\overline{\sigma}_{33} - \overline{\sigma}_{23}\overline{\sigma}_{31}) + x_2(\overline{\sigma}_{13}\overline{\sigma}_{31} - \overline{\sigma}_{11}\overline{\sigma}_{33}) + (x_3\overline{\sigma}_{11}\overline{\sigma}_{23} - \overline{\sigma}_{13}\overline{\sigma}_{21}) \\ x_1(\overline{\sigma}_{22}\overline{\sigma}_{31} - \overline{\sigma}_{21}\overline{\sigma}_{32}) + x_2(\overline{\sigma}_{11}\overline{\sigma}_{32} - \overline{\sigma}_{12}\overline{\sigma}_{31}) + x_3(\overline{\sigma}_{12}\overline{\sigma}_{21} - \overline{\sigma}_{11}\overline{\sigma}_{22}) \end{bmatrix}, \end{aligned} \tag{15}$$

where

$$c = \overline{\sigma}_{13}(\overline{\sigma}_{22}\overline{\sigma}_{31} - \overline{\sigma}_{21}\overline{\sigma}_{32}) + \overline{\sigma}_{12}(\overline{\sigma}_{21}\overline{\sigma}_{33} - \overline{\sigma}_{23}\overline{\sigma}_{31}) + \overline{\sigma}_{11}(\overline{\sigma}_{23}\overline{\sigma}_{32} - \overline{\sigma}_{22}\overline{\sigma}_{33}),$$

and $x_i = \mu_i - r$ is the excess return on asset i.

A.3 Optimal portfolios

Agent $\mathcal{A}$ is unconstrained, so her optimal portfolio proportions are given by

$$\pi_{\mathcal{A},t} = \Sigma_t^{-1}(\mu_t - r\mathbf{1}). \tag{16}$$

Under the market basis the covariance matrix is $\Sigma_t = \underline{\sigma}_t \underline{\sigma}_t'$, so

$$\pi_{\mathcal{A}} = \begin{bmatrix} \frac{\phi_1(1-\rho_{23}^2)-\phi_2(\rho_{12}-\rho_{13}\rho_{23})-\phi_3(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \\ \frac{\phi_2(1-\rho_{13}^2)-\phi_1(\rho_{12}-\rho_{13}\rho_{23})-\phi_3(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \\ \frac{\phi_3(1-\rho_{12}^2)-\phi_1(\rho_{13}-\rho_{12}\rho_{23})-\phi_2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_3(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \end{bmatrix}. \tag{17}$$

As for agent $\mathcal{I}$, I know from Cvitanić and Karatzas (1992) that $\pi_{\mathcal{I},t}$ coincides with the optimal portfolio in the incomplete market:

$$\pi_{\mathcal{I}} = \begin{bmatrix} \pi_{\mathcal{I}}^I \omega_1^I \\ \pi_{\mathcal{I}}^I \omega_2^I \\ 0 \end{bmatrix}, \tag{18}$$

where $\pi_{\mathcal{I},t}^I = (\mu_{I,t} - r)/\sigma_{I,t}^2$, so

$$\pi_{\mathcal{I}} = \begin{bmatrix} \omega_1^I \frac{\phi_I}{\sigma_I} \\ \omega_2^I \frac{\phi_I}{\sigma_I} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\omega_1^I\left(x_1\omega_1^I+x_2\omega_2^I\right)}{\sigma_1^2(\omega_1^I)^2+2\rho_{12}\sigma_1\sigma_2\omega_1^I\omega_2^I+\sigma_2^2(\omega_2^I)^2} \\ \frac{\omega_2^I\left(x_1\omega_1^I+x_2\omega_2^I\right)}{\sigma_1^2(\omega_1^I)^2+2\rho_{12}\sigma_1\sigma_2\omega_1^I\omega_2^I+\sigma_2^2(\omega_2^I)^2} \\ 0 \end{bmatrix}. \tag{19}$$

A.4 Proof of Proposition 1

The market clearing condition imposes that:

$$\omega_t = \pi_{\mathcal{A},t}\nu_{\mathcal{A},t} + \pi_{\mathcal{I},t}\nu_{\mathcal{I},t}$$
$$= \begin{bmatrix} \frac{\nu_{\mathcal{A}}\left(x_3(\rho_{13}-\rho_{12}\rho_{23})\sigma_1\sigma_2+\left(x_2(\rho_{12}-\rho_{13}\rho_{23})\sigma_1+x_1\left(-1+\rho_{23}^2\right)\sigma_2\right)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1^2\sigma_2\sigma_3} - \frac{(-1+\nu_{\mathcal{A}})\omega_1(x_1\omega_1+x_2\omega_2)}{\sigma_1^2\omega_1^2+2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2+\sigma_2^2\omega_2^2} \\ \frac{\nu_{\mathcal{A}}\left(x_3(-\rho_{12}\rho_{13}+\rho_{23})\sigma_1\sigma_2+\left(x_2\left(-1+\rho_{13}^2\right)\sigma_1+x_1(\rho_{12}-\rho_{13}\rho_{23})\sigma_2\right)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1\sigma_2^2\sigma_3} - \frac{(-1+\nu_{\mathcal{A}})\omega_2(x_1\omega_1+x_2\omega_2)}{\sigma_1^2\omega_1^2+2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2+\sigma_2^2\omega_2^2} \\ \frac{\nu_{\mathcal{A}}\left(x_3\left(-1+\rho_{12}^2\right)\sigma_1\sigma_2+(x_2(-\rho_{12}\rho_{13}+\rho_{23})\sigma_1+x_1(\rho_{13}-\rho_{12}\rho_{23})\sigma_2)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1\sigma_2\sigma_3^2} \end{bmatrix}, \tag{20}$$

where $x_i = \mu_i - r$ are excess returns. Solving for x_1, x_2 and x_3, I get:

$$\begin{aligned} x_1^* = &\Big(\sigma_1\left(\sigma_2\sigma_3\omega_2\omega_3\left(\rho_{12}\rho_{13}\sigma_1\omega_1 - \rho_{23}\sigma_1\omega_1 + \rho_{13}\sigma_2\omega_2 - \rho_{12}\rho_{23}\sigma_2\omega_2\right)\right. \\ &\quad \left. +\nu_{\mathcal{A}}\left(\sigma_1\omega_1 + \rho_{12}\sigma_2\omega_2\right)\left(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \rho_{13}\sigma_1\sigma_3\omega_1\omega_3 + \sigma_2^2\omega_2^2 + \rho_{23}\sigma_2\sigma_3\omega_2\omega_3\right)\right)\Big) \\ &\quad / \left(\nu_{\mathcal{A}}\left(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2\right)\right) \\ = &(\omega_1\sigma_1 + \omega_2\rho_{12}\sigma_2)\left(1 - \frac{\omega_3\sigma_3}{\sigma_I^2}(\omega_1\rho_{13}\sigma_1 + \omega_2\rho_{23}\sigma_2\right) \\ &+ \frac{\omega_2\omega_3\sigma_2\sigma_3}{\nu_{\mathcal{A}}\sigma_I^2}\left[\omega_1\sigma_1(\rho_{12}\rho_{13} - \rho_{23}) - \omega_2\sigma_2(\rho_{12}\rho_{23} - \rho_{13})\right]. \end{aligned} \tag{21}$$

I can also write x_1^* in terms of x_I^*:

$$x_1^* = \frac{1}{\nu_A \sigma_I^2 \omega_I^2} \{\sigma_1 (\sigma_2\sigma_3\omega_2\omega_3 (\rho_{12}\rho_{13}\sigma_1\omega_1 - \rho_{23}\sigma_1\omega_1 + \rho_{13}\sigma_2\omega_2 - \rho_{12}\rho_{23}\sigma_2\omega_2)$$
$$+\nu_{\mathcal{A}} (\sigma_1\omega_1 + \rho_{12}\sigma_2\omega_2) (x_I^*\omega_I))\}$$
$$= \frac{1}{\sigma_I^2\omega_I} [x_I^*(\sigma_1\omega_1 + \rho_{1,2}\sigma_2\omega_2)$$
$$+ \frac{\omega_2\omega_3}{\nu_A} \left(\frac{\omega_1}{\omega_I}[cov(R_1, R_2)cov(R_1, R_3) - \sigma_1^2 cov(R_2, R_3)]\right.$$
$$\left.\left. -\frac{\omega_2}{\omega_I}[cov(R_1, R_2)cov(R_2, R_3) - \sigma_2^2 cov(R_1, R_3)]\right)\right]. \tag{22}$$

For x_3^*, I get

$$x_3^* = \Big(\sigma_3 \Big(\nu_{\mathcal{A}}\rho_{13}\sigma_1^3\omega_1^3 + \sigma_1^2\omega_1^2 \Big(\nu_{\mathcal{A}} (2\rho_{12}\rho_{13} + \rho_{23}) \sigma_2\omega_2 + \Big(1 + (-1 + \nu_{\mathcal{A}}) \rho_{13}^2\Big) \sigma_3\omega_3\Big)$$
$$+\sigma_2^2\omega_2^2 \Big(\nu_{\mathcal{A}}\rho_{23}\sigma_2\omega_2 + \Big(1 + (-1 + \nu_{\mathcal{A}}) \rho_{23}^2\Big) \sigma_3\omega_3\Big)$$
$$+\sigma_1\sigma_2\omega_1\omega_2 (2\rho_{12} (\nu_{\mathcal{A}}\rho_{23}\sigma_2\omega_2 + \sigma_3\omega_3) + \rho_{13} (\nu_{\mathcal{A}}\sigma_2\omega_2 + 2 (-1 + \nu_{\mathcal{A}}) \rho_{23}\sigma_3\omega_3))))$$
$$/ \Big(\nu_{\mathcal{A}} \Big(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2\Big)\Big)$$
$$= \omega_I cov(R_I, R_3) + \omega_3\sigma_3^2 \left[1 + \frac{\nu_I}{\nu_A}(1 - \rho_{I,3}^2)\right], \tag{23}$$

where

$$x_I^* = \frac{\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2 + \rho_{13}\sigma_1\sigma_3\omega_1\omega_3 + \rho_{23}\sigma_2\sigma_3\omega_2\omega_3}{\omega_1 + \omega_2}$$
$$= \sigma_I^2\omega_I + \omega_3 cov(R_I, R_3), \tag{24}$$

with $\omega_I = \omega_1 + \omega_2$. Results for x_2 are omitted as they are symmetric to x_1.

A.5 Proof of Proposition 2

Following Cuoco and He (1994), I can still use a social planner to derive equilibrium prices, but the weight λ_t will be stochastic:

$$U_t = E_t \int_t^{\infty} e^{-\delta(s-t)} \left(\log c_{\mathcal{A},s} + \lambda_s \log c_{\mathcal{I},s}\right) ds. \tag{25}$$

The consumption sharing rule is given by:

$$1 = \frac{c_{\mathcal{A},t}^{-1}}{\lambda_t c_{\mathcal{I},t}^{-1}}. \tag{26}$$

I define Agent j's equilibrium share of world consumption as $\nu_{j,t} = \frac{c_{j,t}}{D_{M,t}}$. In equilibrium the two agents must consume the aggregate dividend: $c_{\mathcal{A},t} + c_{\mathcal{I},t} = D_{M,t}$. Thus,

$$\nu_{\mathcal{A},t} = \frac{1}{1+\lambda_t}, \qquad \nu_{\mathcal{I},t} = \frac{\lambda_t}{1+\lambda_t}. \tag{27}$$

As in Basak and Cuoco (1998), the equilibrium state-price density ξ_t is given by the state-price density of the unconstrained agent $\mathcal{A}$:

$$\xi_t = \xi_{\mathcal{A},t} = \kappa_{\mathcal{A}} e^{-\delta t} (\nu_{\mathcal{A},t} D_{M,t})^{-1}. \tag{28}$$

To solve for equilibrium prices, I need to derive an expression λ_t and the related process $\nu_{\mathcal{A},t}$. Sustituting $c_{\mathcal{A}}$ and $c_{\mathcal{I}}$ from (8) in (26), I get:

$$\lambda_t = \frac{\kappa_{\mathcal{A}} \xi_{\mathcal{A},t} / \xi_{\mathcal{A},0}}{\kappa_{\mathcal{I}} \xi_{\mathcal{I},t} / \xi_{\mathcal{I},0}}. \tag{29}$$

Solving (10), agent j's state-price density under the dividend basis, gives:

$$\xi_{j,t} = \xi_{j,0} e^{-\int_0^t (r_s + \frac{1}{2}\theta_{j,s}^2) ds - \int_0^t \overline{\theta}'_{j,s} d\overline{Z}_{D,s}} \tag{30}$$

where $\theta_{j,s} = \overline{\theta}'_{j,s}\mathbf{1}$ and $\mathbf{1}$ is a vector of ones. Substituting (30) in (29) gives:

$$\lambda_t = \frac{\kappa_{\mathcal{A}}}{\kappa_{\mathcal{I}}} e^{-\int_0^t \frac{1}{2}(\theta^2_{\mathcal{A},s} - \theta^2_{\mathcal{I},s})ds - \int_0^t (\overline{\theta}_{\mathcal{A},s} - \overline{\theta}_{\mathcal{I},s})' d\overline{Z}_{D,s}}. \tag{31}$$

Applying Itô's Lemma gives:

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t}dt + \overline{\sigma}'_{\lambda,t}d\overline{Z}_{D,t}, \tag{32}$$

where

$$\mu_{\lambda,t} = \overline{\theta}'_{\mathcal{I},t}(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t}), \tag{33}$$

$$\overline{\sigma}_{\lambda,t} = (\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t}). \tag{34}$$

Rewriting as a scalar process, I get:

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t}dt + \sigma_{\lambda,t}dZ_{\lambda,t}, \tag{35}$$

where

$$\sigma_{\lambda,t} = \sqrt{(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t})'(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t})}, \tag{36}$$

$$dZ_{\lambda,t} = \sigma^{-1}_{\lambda,t}\overline{\sigma}'_{\lambda,t}d\overline{Z}_{D,t}. \tag{37}$$

Remember that:

$$\overline{\theta}_{\mathcal{I}} = \frac{x_I}{\sigma_I^2}\overline{\sigma}' \begin{bmatrix} \omega_1^I \\ \omega_2^I \\ 0 \end{bmatrix}, \qquad \overline{\theta}_{\mathcal{A}} = \overline{\sigma}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore,

$$\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}} = \frac{x_I}{\sigma_I^2}\overline{\sigma}' \begin{bmatrix} \omega_1^I \\ \omega_2^I \\ 0 \end{bmatrix} - \overline{\sigma}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{38}$$

$$\begin{aligned} \overline{\sigma}(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) &= \frac{x_I}{\sigma_I^2}\Sigma \begin{bmatrix} \omega_1^I \\ \omega_2^I \\ 0 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\omega_2^I}{\sigma_I^2}[x_2\omega_2^I(\omega_1^I\sigma_1^2 + \omega_2^I\rho_{12}\sigma_1\sigma_2) - x_1\omega_2^I(\omega_2^I\sigma_2^2 + \omega_1^I\rho_{12}\sigma_1\sigma_2)] \\ \frac{\omega_1^I}{\sigma_I^2}[x_1\omega_1^I(\omega_2^I\sigma_2^2 + \omega_1^I\rho_{12}\sigma_1\sigma_2) - x_2\omega_1^I(\omega_1^I\sigma_1^2 + \omega_2^I\rho_{12}\sigma_1\sigma_2)] \\ x_I\beta_{I,3} - x_3 \end{bmatrix}, \end{aligned} \tag{39}$$

where $\beta_{I,3} = \rho_{I,3}\sigma_3/\sigma_I = (\omega_1^I\rho_{13}\sigma_1\sigma_3 + \omega_2^I\rho_{23}\sigma_2\sigma_3)/\sigma_I^2$. One can easily see that $\overline{\theta}_{\mathcal{I}}'\overline{\theta}_{\mathcal{I}} = \frac{x_I^2}{\sigma_I^2}$ and that $\overline{\theta}_{\mathcal{I}}'\overline{\theta}_{\mathcal{A}} = \frac{x_I^2}{\sigma_I^2}$. Note that those results are basis invariant. I obtain:

$$\mu_\lambda = \overline{\theta}_{\mathcal{I}}'(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) = 0. \tag{40}$$

Similarly,

$$\begin{aligned} \sigma_\lambda^2 &= (\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}})'(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) \\ &= -\frac{x_I^2}{\sigma_I^2} + \overline{\theta}_{\mathcal{A}}'\overline{\theta}_{\mathcal{A}}, \end{aligned} \tag{41}$$

$$\Rightarrow \sigma_\lambda = \sqrt{[x_1\ x_2\ x_3]\Sigma^{-1}[x_1\ x_2\ x_3]' - \frac{x_I^2}{\sigma_I^2}}. \tag{42}$$

Using the definition of $\nu_{\mathcal{A}}$ in (27) and applying Itô Lemma gives:

$$d\nu_{\mathcal{A}} = \mu_{\nu_{\mathcal{A}}}dt + \overline{\sigma}_{\nu_{\mathcal{A}}}'d\overline{Z}_D, \tag{43}$$

where

$$\mu_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}} \nu_{\mathcal{I}}^2 \sigma_\lambda^2, \tag{44}$$

$$\overline{\sigma}_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}} \nu_{\mathcal{I}} \overline{\sigma}_\lambda. \tag{45}$$

In scalar notation this becomes:

$$d\nu_{\mathcal{A}} = \mu_{\nu_{\mathcal{A}}} dt + \sigma_{\nu_{\mathcal{A}}} dZ_\lambda, \tag{46}$$

$$\sigma_{\nu_{\mathcal{A}}} = -\nu_{\mathcal{A}} \nu_{\mathcal{I}} \sigma_\lambda. \tag{47}$$

Applying Itô's Lemma to (28), I obtain:

$$\begin{aligned} \frac{d\xi}{\xi} = &- \left[\delta + \mu_{D_M} - \sigma_{D_M}^2 + \frac{\rho_{\nu_{\mathcal{A}} D_M} \sigma_{\nu_{\mathcal{A}}} \sigma_{D_M}}{\nu_{\mathcal{A}}} \right] dt \\ &- \left[\overline{\sigma}'_{D_M} + \frac{\overline{\sigma}'_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}} \right] d\overline{Z}_D. \end{aligned} \tag{48}$$

Equaling the terms to those in (10), I get:

$$r_f = \delta + \mu_{D_M} - \sigma_{D_M}^2 + \frac{\rho_{\nu_{\mathcal{A}} D_M} \sigma_{\nu_{\mathcal{A}}} \sigma_{D_M}}{\nu_{\mathcal{A}}}, \tag{49}$$

$$\overline{\theta} = \overline{\sigma}_{D_M} + \frac{\overline{\sigma}_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}}. \tag{50}$$

A.6 Proof of Corollary 1

From (28) I can assert that $\overline{\theta} = \overline{\theta}_{\mathcal{A}}$. Thus, from (34), (45) and (50),

$$\begin{aligned}
\overline{\theta}_{\mathcal{A}} &= \overline{\sigma}_{D_M} + \frac{\overline{\sigma}_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}} \\
&= \overline{\sigma}_{D_M} - \nu_{\mathcal{I}} \overline{\sigma}_{\lambda} \\
&= \overline{\sigma}_{D_M} - \nu_{\mathcal{I}} (\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}), \qquad (51) \\
\Rightarrow \overline{\theta} &= \frac{\overline{\sigma}_{D_M}}{\nu_{\mathcal{A}}} - \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \overline{\theta}_{\mathcal{I}}, \qquad (52) \\
\overline{\theta} &= \overline{\sigma}_{D_M} + \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\sigma}_{D_M} - \overline{\theta}_{\mathcal{I}} \right). \qquad (53)
\end{aligned}$$

Note here that $\overline{\sigma}_{D_M}$ is exogenous to the model (when defined relative to the dividend basis), $\nu_{\mathcal{I}}$ and $\nu_{\mathcal{A}} = 1 - \nu_{\mathcal{I}}$ are state variables and the other quantities are determined endogenously in equilibrium. Denoting $\overline{\theta}^{\star} = \overline{\sigma}_{D_M}$ the price of risk when there are no indexers ($\nu_{\mathcal{A}} = 1$, $\nu_{\mathcal{I}} = 0$),

$$\begin{aligned}
\overline{\theta} &= \overline{\theta}^{\star} + \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{I}} \right) \\
\Rightarrow \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{A}} \right) &= -\frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{I}} \right). \qquad (54)
\end{aligned}$$

A.7 Proof of Proposition 3

In this section I derive the dynamics of each stock's price process. The price $S_{i,t}$ of stock i at time t is the expected value of future dividends discounted using the stochastic discount factor of the representative agent ξ defined in (48):

$$S_{i,t} = E_t \left[\int_t^{\infty} \frac{\xi_{\tau}}{\xi_t} D_{i,\tau} d\tau \right]. \qquad (55)$$

Using the results from equations (8) and (28), I have

$$S_{i,t} = E_t \left[\int_t^{\infty} e^{-\delta(\tau - t)} \left(\frac{c_{\mathcal{A},\tau}}{c_{\mathcal{A},t}} \right)^{-1} D_{i,\tau} d\tau \right]. \tag{56}$$

From (27), I have:

$$c_{\mathcal{A},t} = \frac{D_{M,t}}{1 + \lambda_t}, \tag{57}$$

thus

$$\frac{c_{\mathcal{A},t}}{c_{\mathcal{A},\tau}} = \frac{D_{M,t}}{D_{M,\tau}} \frac{1 + \lambda_\tau}{1 + \lambda_t}. \tag{58}$$

Substituting this last result in (56), I obtain:

$$\begin{aligned} S_{i,t} &= D_{M,t} E_t \left[\int_t^{\infty} e^{-\delta(\tau - t)} \frac{1 + \lambda_\tau}{1 + \lambda_t} s_{i,\tau} d\tau \right] \\ &= D_{M,t} f_{i,t}, \end{aligned} \tag{59}$$

where

$$\begin{aligned} f_{i,t} &= E_t \left[\int_t^{\infty} e^{-\delta(\tau - t)} \frac{1 + \lambda_\tau}{1 + \lambda_t} s_{i,\tau} d\tau \right] \\ &= \underbrace{\frac{1}{1 + \lambda_t}}_{\nu_{\mathcal{A},t}} \underbrace{E_t \left[\int_t^{\infty} e^{-\delta(\tau - t)} s_{i,\tau} d\tau \right]}_{f_{i,t}^{\mathcal{A}}} + \underbrace{\frac{\lambda_t}{1 + \lambda_t}}_{\nu_{\mathcal{I},t}} \underbrace{E_t \left[\int_t^{\infty} e^{-\delta(\tau - t)} \frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau \right]}_{f_{i,t}^{\mathcal{I}}} \qquad (60) \\ &= \nu_{\mathcal{A},t} f_{i,t}^{\mathcal{A}} + \nu_{\mathcal{I},t} f_{i,t}^{\mathcal{I}}. \qquad (61) \end{aligned}$$

Note that in a world without constraints, λ_t is constant and we thus have $f_{i,t} = f_{i,t}^{\mathcal{A}}$. Alternatively, I can get this result by setting $\nu_{\mathcal{A},t} = 1$ and $\nu_{\mathcal{I},t} = 0$.

A.8 Solving for $f_{i,t}^{\mathcal{A}}$

$f_{i,t}^{\mathcal{A}}$ depends on the relative share of the aggregate dividend of each stock, $s_{i,t}$ as defined in (3). Therefore,

$$s_{i,t} = \frac{D_{i,t}}{D_{M,t}}. \tag{62}$$

To fully characterize the relative weights of each dividend stream two of those s_i are sufficient, so I need two state variables. Using Itô's Lemma, I obtain:

$$\begin{aligned}\frac{ds_i^M}{s_i^M} &= \left[\overline{\sigma}'_{D_M}(\overline{\sigma}_{D_M} - \overline{\sigma}_{D_i})\right] dt \\ &\quad + (\overline{\sigma}_{D_i} - \overline{\sigma}_{D_M})' d\overline{Z}_D, \end{aligned} \tag{63}$$

which after simplification yields

$$ds_i = \mu_{s_i} dt + \overline{\sigma}'_{s_i} d\overline{Z}_D, \tag{64}$$

where

$$\mu_{s_i} = s_i s_{-i} \left[-s_i \sigma_D^2 + s_{-i}\sigma_{D_{-i}}^2 + (s_i - s_{-i})\rho_{D_i D_{-i}} \sigma_D \sigma_{D_{-i}}\right], \tag{65}$$

$$\overline{\sigma}_{s_i} = s_i s_{-i}(\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}), \tag{66}$$

and D_{-i} represents the dividend stream of the other two stocks combined.

Defining $x_{i,t} = \log \frac{s_{i,t}}{s_{-i,t}}$, it follows from Itô's Lemma that:

$$dx_i = \mu_{x_i} dt + \overline{\sigma}'_{x_i} d\overline{Z}_D \tag{67}$$

where

$$\mu_{x_i} = \left[\mu_{D_i} - \frac{1}{2}\sigma^2_{D_i}\right] - \left[\mu_{D_{-i}} - \frac{1}{2}\sigma^2_{D_{-i}}\right], \tag{68}$$

$$\overline{\sigma}_{x_i} = \overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}. \tag{69}$$

In scalar form,

$$dx_i = \mu_{x_i} dt + \sigma_{x_i} dZ_{x_i}, \tag{70}$$

where

$$\begin{aligned} \sigma_{x_i} &= \sqrt{(\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}})'\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}} \\ &= \sqrt{\sigma^2_{D_i} + \sigma^2_{D_{-i}} - 2\rho_{D_i D_{-i}}\sigma_{D_i}\sigma_{D_{-i}}}, \end{aligned} \tag{71}$$

$$Z_{x_i} = \sigma_{x_i}^{-1}\overline{\sigma}'_{x_i} d\overline{Z}_D. \tag{72}$$

From Cochrane, Longstaff, and Santa-Clara (2008), I know there is a closed-form expression for $f^{\mathcal{A}}_{i,t}$ if x_i is the only relevant state variable ($\nu_{\mathcal{A}}$ is irrelevant for $f^{\mathcal{A}}_{i,t}$). In the present case the moments of the dividend process of portfolio $-i$ also depend on the relative dividend of the two stocks in that portfolio, i.e. x_1 depends on D_2/D_3. So $f^{\mathcal{A}}_{i,t}$ depends on two state variables representing the relative dividend processes. Let's use x_1 and x_2 as the state variables. Note also that since $\sum_{i=1}^{3} f^{\mathcal{A}}_{i,t} = \frac{1}{\delta}$, we only need to solve for two i to get the third one. We'll solve for $i = 1, 2$ so the functions will be symmetric. Here I show the derivation of $f^{\mathcal{A}}_{1,t}$. Note from (64) that $s_i = 0$ and $s_i = 1$

are absorbing states, so we obtain the following boundary conditions:

$$\lim_{x_1 \to -\infty} f_{1,t}^{\mathcal{A}} = 0, \tag{73}$$

$$\lim_{x_1 \to \infty} f_{1,t}^{\mathcal{A}} = \frac{1}{\delta}, \tag{74}$$

$$\lim_{x_2 \to \infty} f_{1,t}^{\mathcal{A}} = 0. \tag{75}$$

The boundary condition $\lim_{x_2 \to -\infty} f_{1,t}^{\mathcal{A}}$ is less obvious because in that case asset 2 becomes irrelevant, so $f_{1,t}^{\mathcal{A}}$ converges to the Cochrane, Longstaff, and Santa-Clara (2008) case. From the Feynman-Kac theorem, we can transform the problem to a PDE representation:

$$\frac{1}{2}\sigma_{x_1}^2 \frac{\partial^2 f_1^{\mathcal{A}}}{\partial x_1^2} + \frac{1}{2}\sigma_{x_2}^2 \frac{\partial^2 f_1^{\mathcal{A}}}{\partial x_2^2} + \overline{\sigma}_{x_1}' \overline{\sigma}_{x_2} \frac{\partial^2 f_1^{\mathcal{A}}}{\partial x_1 \partial x_2} + \mu_{x_1} \frac{\partial f_1^{\mathcal{A}}}{\partial x_1} + \mu_{x_2} \frac{\partial f_1^{\mathcal{A}}}{\partial x_2} - \rho f_1^{\mathcal{A}} + \frac{1}{1 + e^{-x_1}} = 0, \tag{76}$$

where

$$\begin{aligned}
\mu_{x_1} &= -\left[\frac{s_2 - s_1 s_2 - s_2^2}{(1 - s_1)^2}\right](1 - \rho_D)\sigma_D^2, \\
\mu_{x_2} &= -\left[\frac{s_1 - s_1 s_2 - s_1^2}{(1 - s_2)^2}\right](1 - \rho_D)\sigma_D^2, \\
\sigma_{x_1}^2 &= \left[2 - \frac{2(s_2 - s_1 s_2 - s_2^2)}{(1 - s_1)^2}\right](1 - \rho_D)\sigma_D^2, \\
\sigma_{x_2}^2 &= \left[2 - \frac{2(s_1 - s_1 s_2 - s_1^2)}{(1 - s_2)^2}\right](1 - \rho_D)\sigma_D^2, \\
\overline{\sigma}_{x_1}' \overline{\sigma}_{x_2} &= \left[\frac{(1 + s_1(-3 + 2s_1) - 3s_2 + 2s_1 s_2 + 2s_2^2)}{(1 - s_1)(1 - s_2)}\right](1 - \rho_D)\,\sigma_D^2.
\end{aligned}$$

Following Bhamra (2007), I use a perturbation expansion of the form:

$$f_1^{\mathcal{A}} = f_{1,0}^{\mathcal{A}} + \epsilon f_{1,1}^{\mathcal{A}} + \epsilon^2 f_{1,2}^{\mathcal{A}} + \ldots \tag{77}$$

Defining $\rho_D = 1 - 2\epsilon^2$, I get:

$$f^{\mathcal{A}}_{1,0} = \frac{1}{\delta + e^{-x_1}\delta},$$
$$f^{\mathcal{A}}_{1,1} = 0,$$
$$f^{\mathcal{A}}_{1,2} = \frac{e^{x_1}\left(1 - e^{x_1}\left(-1+s_1\right)^2 + s_1^2 + 2s_1\left(-1+s_2\right) + 2\left(-1+s_2\right)s_2\right)\sigma_D^2}{\left(1+e^{x_1}\right)^3\left(-1+s_1\right)^2\delta^2},$$
$$f^{\mathcal{A}}_{1,3} = 0.$$

After simplification, I obtain:

$$f^{\mathcal{A}}_1 = \frac{s_1}{\delta} - \frac{s_1\left(1 - 3s_1 + 2s_1^2 - 2s_2 + 2s_1s_2 + 2s_2^2\right)\left(-1+\rho_D\right)\sigma_D^2}{2\delta^2} + O(\epsilon^4). \tag{78}$$

A.9 Solving for $f^{\mathcal{I}}_{i,t}$

Remember that

$$f^{\mathcal{I}}_{i,t} = E_t\left[\int_t^\infty e^{-\delta(\tau-t)}\frac{\lambda_\tau}{\lambda_t}s_{i,\tau}d\tau\right], \tag{79}$$

which depends on $x_{1,t}$, $x_{2,t}$ and $\nu_{A,t} = \frac{1}{1+\lambda_t}$. Note that λ is a local martingale and that assuming σ_λ is bounded, then it is an exponential martingale. I can then define a new measure:[2]

$$\mathbb{P}'(A_T) = E_t\left[1_{A_T}\lambda_T\right], \quad \forall t, \quad T \in [0,\infty) \quad t \leq T. \tag{80}$$

With this change of measure,

$$f^{\mathcal{I}}_{i,t} = E^{\mathbb{P}'}_t\left[\int_t^\infty e^{-\delta(\tau-t)}s_{i,\tau}d\tau\right]. \tag{81}$$

[2] See pages 28-29 of Karatzas and Shreve (1998) for details.

From (81), it follows that $f_{i,t}^{\mathcal{I}}$ satisfies a BSDE. The coefficients of the BSDE will depend on $\nu_{i,t}^{\mathcal{A}}$, which satisfies a FSDE. Together they form a FBSDE. The Feynman-Kac theorem still applies thus $f_{i,t}^{\mathcal{I}}$ satisfies the following inhomogeneous elliptic PDE:

$$\begin{aligned}\mu_{x_1}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial x_1}+\mu_{x_2}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial x_2}+\mu_{\nu_{\mathcal{A}}}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial \nu_{\mathcal{A}}}+\frac{1}{2}\sigma_{x_1}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial x_1^2}+\frac{1}{2}\sigma_{x_2}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial x_2^2}+\frac{1}{2}\sigma_{\nu_{\mathcal{A}}}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial \nu_{\mathcal{A}}^2}\\+\overline{\sigma}_{x_1}'\overline{\sigma}_{x_2}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial x_1\partial x_2}+\overline{\sigma}_{x_1}'\overline{\sigma}_{\nu_A}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial x_1\partial \nu_{\mathcal{A}}}+\overline{\sigma}_{x_2}'\overline{\sigma}_{\nu_A}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial x_2\partial \nu_{\mathcal{A}}}-\rho f_1^{\mathcal{I}}+\frac{1}{1+e^{-x_1}}=0,\quad (82)\end{aligned}$$

where

$$\begin{aligned}\mu_{x_1}^{\mathbb{P}'}&=\mu_{x_1}+\overline{\sigma}_{x_1}'\overline{\sigma}_\lambda,\\ \mu_{x_2}^{\mathbb{P}'}&=\mu_{x_2}+\overline{\sigma}_{x_2}'\overline{\sigma}_\lambda,\\ \mu_{\nu_{\mathcal{A}}}^{\mathbb{P}'}&=\mu_{\nu_{\mathcal{A}}}+\overline{\sigma}_{\nu_{\mathcal{A}}}'\overline{\sigma}_\lambda\\ &=-\nu_{\mathcal{A}}^2(1-\nu_{\mathcal{A}})\sigma_\lambda^2,\\ \overline{\sigma}_{x_1}'\overline{\sigma}_{\nu_A}&=-\nu_{\mathcal{A}}(1-\nu_{\mathcal{A}})\overline{\sigma}_{x_1}'\overline{\sigma}_\lambda,\\ \overline{\sigma}_{x_2}'\overline{\sigma}_{\nu_A}&=-\nu_{\mathcal{A}}(1-\nu_{\mathcal{A}})\overline{\sigma}_{x_2}'\overline{\sigma}_\lambda,\\ \overline{\sigma}_{x_1}'\overline{\sigma}_\lambda&=\overline{\sigma}_{D_1}'\overline{\sigma}_\lambda-\left(\frac{s_2}{1-s_1}\right)\overline{\sigma}_{D_2}'\overline{\sigma}_\lambda-\left(1-\frac{s_2}{1-s_1}\right)\overline{\sigma}_{D_3}'\overline{\sigma}_\lambda,\\ \overline{\sigma}_{x_2}'\overline{\sigma}_\lambda&=\overline{\sigma}_{D_2}'\overline{\sigma}_\lambda-\left(\frac{s_1}{1-s_2}\right)\overline{\sigma}_{D_1}'\overline{\sigma}_\lambda-\left(1-\frac{s_1}{1-s_2}\right)\overline{\sigma}_{D_3}'\overline{\sigma}_\lambda.\end{aligned}$$

Note that $\sigma_{\nu_{\mathcal{A}}}^2$ also depends on σ_λ^2 and that $\overline{\sigma}_\lambda$ (and σ_λ^2) depends on the endogenously determined $\overline{\sigma}$.

A.9.1 Boundary conditions

The required boundary conditions are the following:

$$\lim_{x_1 \to -\infty} f^{\mathcal{I}}_{1,t} = 0, \tag{83}$$

$$\lim_{x_1 \to \infty} f^{\mathcal{I}}_{1,t} = \frac{1}{\delta}, \tag{84}$$

$$\lim_{x_2 \to \infty} f^{\mathcal{I}}_{1,t} = 0, \tag{85}$$

$$\lim_{\nu_{\mathcal{A}} \to 1} \nu_{\mathcal{I}} f^{\mathcal{I}}_{1,t} = 0, \tag{86}$$

$$\left. \frac{\partial f^{\mathcal{I}}_{1,t}}{\partial \nu_{\mathcal{A}}} \right|_{\nu_{\mathcal{A}}=0} = 0. \tag{87}$$

Finally, when $x_2 \to -\infty$, then the second dividend tree becomes irrelevant and $f^{\mathcal{I}}_{1,t}$ converges to the case of Bhamra (2007). The other boundary conditions are justified as follows:

1. $\lim_{x_1 \to -\infty} f^{\mathcal{I}}_{1,t} = 0$ and $\lim_{x_2 \to \infty} f^{\mathcal{I}}_{1,t} = 0$: When $x_2 \to \infty$, I must be that $x_1 \to -\infty$. When $x_1 \to -\infty$, the first dividend stream becomes irrelevant so investors aren't willing to pay anything to own the stock.

2. $\lim_{x_1 \to \infty} f^{\mathcal{I}}_{1,t} = \frac{1}{\delta}$: In this case there is a single dividend tree and complete markets (the constraint becomes irrelevant), so:

$$S_1 = \frac{D_1}{\delta} = D_M(\nu_{\mathcal{A},t} f^{\mathcal{A}}_{1,t} + \nu_{\mathcal{I},t} f^{\mathcal{I}}_{1,t}),$$
$$\Rightarrow \frac{1}{\delta} = \nu_{\mathcal{A},t}\left(\frac{1}{\delta}\right) + (1 - \nu_{\mathcal{A},t}) f^{\mathcal{I}}_{1,t} = f^{\mathcal{I}}_{1,t}.$$

3. $\lim_{\nu_{\mathcal{A}} \to 1} \nu_{\mathcal{I}} f^{\mathcal{I}}_{1,t} = 0$: When $\nu_{\mathcal{A}} = 1$, agent $\mathcal{A}$, which faces no constraint, consumes all dividends so markets are complete. Therefore $f_{1,t}\big|_{\nu_{\mathcal{A}}=1} = f^{\mathcal{A}}_{1,t}$ so this boundary

condition must hold.

4. $\left.\frac{\partial f^{\mathcal{I}}_{1,t}}{\partial \nu_{\mathcal{A}}}\right|_{\nu_{\mathcal{A}}=0} = 0$: As $\nu_{\mathcal{A}} \to 0$, indexers consume all dividends. However, they have a worst investment opportunity set than active investors, so this can't hold for more than an instant. Therefore this boundary condition must be a reflecting boundary condition.

A.10 Matching moments

I now have expressions for both $f^{\mathcal{A}}_{i,t}$ and $f^{\mathcal{I}}_{i,t}$. I have a closed form expression for $f^{\mathcal{A}}_{i,t}$ that depends on exogenous parameters and state variables, which is easy to evaluate numerically. For $f^{\mathcal{I}}_{i,t}$, I have a PDE that can be approximated. However, the current form of that solution depends on the endogenously determined $\overline{\sigma}$ because of the dependence on $\overline{\sigma}_{\lambda}$. I have that $S_{i,t} = D_{M,t} f_{i,t}$, so:

$$\begin{aligned} dS_i &= D_M df_i + f_i dD_M + df_i dD_M, \\ \frac{dS_i}{S_i} &= \frac{df_i}{f_i} + \frac{dD_M}{D_M} + \frac{df_i}{f_i}\frac{dD_M}{D_M}, \end{aligned} \tag{88}$$

where

$$\frac{dD_M}{D_M} = \mu_D dt + \overline{\sigma}'_D d\overline{Z},$$

and $\overline{\sigma}_D = s_1\overline{\sigma}_{D_1} + s_2\overline{\sigma}_{D_2} + (1 - s_1 - s_2)\overline{\sigma}_{D_3}$. I know that $f_{i,t}$ is a function of exogenous parameters and state processes s_1, s_2 and $\nu_{\mathcal{A}}$, therefore from Itô's Lemma I get:

$$
\begin{aligned}
df_i = &\left[\mu_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \mu_{s_1}\frac{\partial f_i}{\partial s_1} + \mu_{s_2}\frac{\partial f_i}{\partial s_2} + \frac{1}{2}\left(\sigma^2_{\nu_{\mathcal{A}}}\frac{\partial^2 f_i}{\partial \nu^2_{\mathcal{A}}} + \sigma^2_{s_1}\frac{\partial^2 f_i}{\partial s_1^2} + \sigma^2_{s_2}\frac{\partial^2 f_i}{\partial s_2^2}\right.\right. \\
&\left.\left. +2\overline{\sigma}'_{\nu_{\mathcal{A}}}\overline{\sigma}_{s_1}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_1} + 2\overline{\sigma}'_{\nu_{\mathcal{A}}}\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_2} + 2\overline{\sigma}'_{s_1}\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial s_1\partial s_2}\right)\right] dt \\
&+ \left[\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1}\frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2}\frac{\partial f_i}{\partial s_2}\right]' d\overline{Z}. \qquad (89)
\end{aligned}
$$

From the definition of stock return process, I also have that:

$$
\frac{dS_i}{S_i} = \left[\mu_i - \frac{D_i}{S_i}\right] dt + \overline{\sigma}'_i d\overline{Z}, \qquad (90)
$$

where

$$
\begin{aligned}
\mu_i = &\frac{1}{f_i}\left[\mu_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \mu_{s_1}\frac{\partial f_i}{\partial s_1} + \mu_{s_2}\frac{\partial f_i}{\partial s_2} + \frac{1}{2}\left(\sigma^2_{\nu_{\mathcal{A}}}\frac{\partial^2 f_i}{\partial \nu^2_{\mathcal{A}}} + \sigma^2_{s_1}\frac{\partial^2 f_i}{\partial s_1^2} + \sigma^2_{s_2}\frac{\partial^2 f_i}{\partial s_2^2}\right.\right. \\
&\left. +2\overline{\sigma}'_{\nu_{\mathcal{A}}}\overline{\sigma}_{s_1}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_1} + 2\overline{\sigma}'_{\nu_{\mathcal{A}}}\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_2} + 2\overline{\sigma}'_{s_1}\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial s_1\partial s_2}\right) \\
&\left. + \left(\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1}\frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2}\frac{\partial f_i}{\partial s_2}\right)' \overline{\sigma}_D\right] + \mu_D, \qquad (91) \\
\overline{\sigma}_i = &\frac{1}{f_i}\left[\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1}\frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2}\frac{\partial f_i}{\partial s_2}\right] + \overline{\sigma}_D. \qquad (92)
\end{aligned}
$$

Note that the expression I have for $\overline{\sigma}_\lambda$ from (34) is a function of both $\overline{\sigma}$ and the equilibrium price ratio f_1/f_2, since $\omega^I_1 = 1 + f_1/f_2$ and $\omega^I_2 = 1 + f_2/f_1$. I first use the definitions of $\overline{\sigma}$ and $\overline{\sigma}_\lambda$ to create perturbation expansions of these moments as a function of f_1, f_2, f_3 and their own expansions. Substituting these expansions in the PDE (82), I create a perturbation expansion of the PDE, and then solve by equating terms in the different powers of ϵ. The result is the closed-form approximation

$$f_1^{\mathcal{I}} = f_1^{\mathcal{A}} + \frac{1}{2\left(s_1+s_2\right)\nu_{\mathcal{A}}\delta^2} s_1 \Big(2\left(-1+s_1+s_2\right)\Big(-s_2+2\left(s_1^2+s_1\left(-1+s_2\right)+s_2^2\right)\Big) \\ +\left(s_1+s_2\right)\Big(1+2s_1^2+2\left(-1+s_2\right)s_2+s_1\left(-3+2s_2\right)\Big)\nu_{\mathcal{A}}\Big)\left(1-\rho_D\right)\sigma_D^2 + O(\epsilon^4). \tag{93}$$

As in the unconstrained economy, I find $f_2^{\mathcal{I}}$ by symmetry and $f_3^{\mathcal{A}}$ by $f_3^{\mathcal{I}} = \frac{1}{\delta} - f_1^{\mathcal{I}} - f_2^{\mathcal{I}}$. A drawback of the use of a perturbation expansion is that it is impossible to guarantee that the boundary conditions will be satisfied. It is easy to see that in this case (87) is not satisfied, which means that the approximation will not be valid in the neighbourhood of $\nu_{\mathcal{A}} = 0$. Since this region is not economically important for the current analysis,[3] this does not pose a problem as long as the analysis focuses on values of $\nu_{\mathcal{A}}$ that are away from that boundary.

B Vector notation

This section introduces the two different vector bases I use in the proofs. While not a necessary read, this section is a useful appendix for understanding the proofs. The reason for using different bases is to simplify certain steps of the proof. Steps involving stock returns are easier to solve under the market basis. However, when solving for equilibrium stock return dynamics, the dividend basis is more appropriate. The

[3] $\nu_{\mathcal{A}} = 0$ corresponds to the case where the aggregate wealth is fully owned by the indexer, and the remaining active investor still has to hold the share of the non-index stock. The realization of such a scenario seems highly unlikely.

dividend processes in (1) can be represented as a vector:

$$\frac{dD_t}{D_t} = \mu_D \mathbf{1} dt + \sigma_D \mathbf{1}' dZ_{D_t}, \tag{94}$$

where $\frac{dD_t}{D_t}$ is a vector with $\frac{dD_{i,t}}{D_{i,t}}$ as the i-th element and dZ_{D_t} is a vector with $dZ_{D_{i,t}}$ as the i-th element. Since the $dZ_{D_{i,t}}$ can be correlated, we can represent the correlation matrix of dZ_{D_t} as

$$C_{D_t} = \begin{bmatrix} 1 & \rho_D & \rho_D \\ \rho_D & 1 & \rho_D \\ \rho_D & \rho_D & 1 \end{bmatrix}.$$

Stock returns in (4) can also be represented in vector notation:

$$dR_t = \mu_t dt + \sigma_t dZ_t,$$

where dR_t, μ_t and dZ_t are vectors with $dR_{i,t}$, $\mu_{i,t}$ and $dZ_{i,t}$ as the i-th element and σ_t is a diagonal matrix with $\sigma_{i,t}$ as the i-th diagonal element. The dZ_t BM are correlated with correlation matrix:

$$C_t = \begin{bmatrix} 1 & \rho_{t,12} & \rho_{t,13} \\ \rho_{t,12} & 1 & \rho_{t,23} \\ \rho_{t,13} & \rho_{t,23} & 1 \end{bmatrix}.$$

B.1 Rotation matrix

It is often easier to deal with independent Brownian motions (BM) than correlated ones. It is possible to transform a multivariate BM to a vector of independent BM using a rotation matrix. Under that transformation, drifts, variances and covariances of Itô processes are invariant. Consider the three-dimensional multivariate BM $Z =$

$[Z_1\ Z_2\ Z_3]'$ with correlation matrix:

$$C = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

Using the Cholesky decomposition, we can construct a rotation matrix K to transform Z into a three-dimensional vector of independent BM. From the Cholesky decomposition, we get the lower triangular matrix L such that $LL' = C$. The matrix L is often used to generate correlated BM from independent ones such that $Z = LX$. In this case, I am interested in the inverse process: $X = KZ$ where $K = L^{-1}$.

Applying the Cholesky decomposition to the matrix C,

$$K = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\rho_{12}}{\sqrt{1-\rho_{12}^2}} & \frac{1}{\sqrt{1-\rho_{12}^2}} & 0 \\ \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{12}^2-\rho_{13}^2+2\rho_{12}\rho_{13}\rho_{23}-\rho_{23}^2)}} & \frac{-\rho_{12}\rho_{13}+\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{12}^2-\rho_{13}^2+2\rho_{12}\rho_{13}\rho_{23}-\rho_{23}^2)}} & \frac{1}{\sqrt{1+\frac{\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2}{-1+\rho_{12}^2}}} \end{bmatrix}. \tag{95}$$

Changing the set of BMs using a rotation matrix is called a change of basis. Drift terms, total variances and covariances between processes are invariant under a change of basis. Note that if the initial BM are uncorrelated (correlation terms in C all equal to 0), then the rotation matrices L and K collapse to the identity matrix.

B.2 Dividend basis

The BM driving the dividend processes described in (94) are correlated. Consider L_{D_t}, the lower triangular matrix from the Cholesky decomposition of C_{D_t}, and it's inverse

K_{D_t}. Then I can rewrite (94) as:

$$\begin{aligned}\frac{dD_t}{D_t} &= \mu_D dt + \sigma_D dZ_{D_t} \\ &= \mu_D dt + \sigma_D L_D \overline{Z}_{D_t} \\ &= \mu_D dt + \overline{\sigma}_D \overline{Z}_{D_t},\end{aligned}$$

where $\overline{\sigma}_D = \sigma_D L_D$ and $\overline{Z}_{D_t} = K_D Z_{D_t}$. This transformation yields a new basis that I call the dividend basis. The variance matrix under the dividend basis can be written as:

$$\overline{\sigma}_D = \begin{pmatrix} 1 & 0 & 0 \\ \rho_D & \sqrt{1-\rho_D^2} & 0 \\ \rho_D & \frac{\sqrt{1-\rho_D}\rho_D}{\sqrt{1+\rho_D}} & \sqrt{3 - 2\rho_D - \frac{2}{1+\rho_D}} \end{pmatrix}. \tag{96}$$

B.3 Market basis

Similarly, the BM driving the market return processes in (4) might be correlated as they are determined endogenously. Consider L_t, the lower triangular matrix from the Cholesky decomposition of C_t, and it's inverse K_t. Then I can write:

$$\begin{aligned}dR_t &= \mu_t dt + \sigma_t dZ_t \\ &= \mu_t dt + \sigma_t L_t d\underline{Z}_t \\ &= \mu_t dt + \underline{\sigma}_t d\underline{Z}_{t,}\end{aligned}$$

where $\underline{\sigma}_t = \sigma_t L_t$ and $\underline{Z}_t = K_t Z_t$. This transformation yields a new basis that I call the market basis. Under this basis,

$$\underline{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ \rho_{12}\sigma_2 & \sqrt{1-\rho_{12}^2}\sigma_2 & 0 \\ \rho_{13}\sigma_3 & \frac{-\rho_{12}\rho_{13}+\rho_{23}}{\sqrt{1-\rho_{12}^2}}\sigma_3 & \sqrt{1 + \frac{\rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23} + \rho_{23}^2}{-1+\rho_{12}^2}}\sigma_3 \end{bmatrix}. \tag{97}$$

Note that the return process can also be written under the dividend basis as:

$$dR_t = \mu_t dt + \overline{\sigma}_t d\overline{Z}_{D_t},$$

where $\overline{\sigma}_t d\overline{Z}_{D_t} = \sigma_t dZ_t = \underline{\sigma}_t d\underline{Z}_t$. $\overline{\sigma}_t$ has the generic form:

$$\overline{\sigma}_t = \begin{bmatrix} \overline{\sigma}_{11} & \overline{\sigma}_{12} & \overline{\sigma}_{13} \\ \overline{\sigma}_{21} & \overline{\sigma}_{22} & \overline{\sigma}_{23} \\ \overline{\sigma}_{31} & \overline{\sigma}_{32} & \overline{\sigma}_{33} \end{bmatrix}. \tag{98}$$

However, this leaves 9 unknowns to solve for in $\overline{\sigma}_t$ (it is a 3×3 matrix), whereas the known structure of $\underline{\sigma}_t$ leaves only 6 unknowns to solve for, namely σ_1, σ_2, σ_3, ρ_{12}, ρ_{13} and ρ_{23}.

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